CONTACT PROBLEMS FOR AN ELASTIC LAYER OF SLIGHT THICKNESS

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Three-dimensional contact problems for an elastic layer of thickness h lying on a rigid base without friction are considered. Friction forces between the stamp and the layer are assumed absent.

The case when the region of contact of the stamp with the layer is an infinite strip of width $2a$ is studied.

The base of the stamp is arbitrary. The whole analysis applies to the case when the relative thickness of the layer $\lambda = h/a$ is relatively small.

The method of [1] is perfected and developed further, and examples are given.

1. Formulation of the contact problem for an elastic layer. The problem of the effect of a stamp in the shape of an infinite strip on an elastic layer of slight thickness reduces to the solution of the system of Lamé equations

$$(1-2\nu)\,\Delta u+\frac{\partial\vartheta}{\partial x}=0,\quad (1-2\nu)\,\Delta v+\frac{\partial\vartheta}{\partial y}=0,\quad (1-2\nu)\,\Delta w+\frac{\partial\vartheta}{\partial z}=0 \tag{1.1}$$

with the boundary conditions

for $z = h$

$$\begin{aligned}
\tau_{xz} &= G\,(\partial u/\partial z+\partial w/\partial x)=0 && (-\infty<x,\,y<\infty)\\
\tau_{yz} &= G\,(\partial v/\partial z+\partial w/\partial y)=0 && (-\infty<x,\,y<\infty)\\
\sigma_z &= 2G\,[\partial w/\partial z+\vartheta\nu/(1-2\nu)]=0 && (|y|>a,\ |x|<\infty)\\
w &= -f(x,\,y) && (|y|\leqslant a,\ |x|<\infty)
\end{aligned} \tag{1.2}$$

for $z = 0$

$$\begin{aligned}
\tau_{xz} &= G\,(\partial u/\partial z+\partial w/\partial x)=0 && (-\infty<x,\,y<\infty)\\
\tau_{yz} &= G\,(\partial v/\partial z+\partial w/\partial y)=0 && (-\infty<x,\,y<\infty)\\
w &= 0 && (-\infty<x,\,y<\infty)
\end{aligned} \tag{1.3}$$

The displacements decrease as $|y| \to \infty$.

Here Δ is the three-dimensional Laplace operator, ν is the Poisson coefficient, G is the shear modulus, $f(x, y)$ is the function of settling of points of the surface of the elastic layer under the stamp, and is even in y.

We seek to determine the contact stresses

$$\sigma_z |_{z=h} = -q(x, y) \qquad (|y| \leqslant a, \ -\infty < x < \infty) \tag{1.4}$$

due to the interaction between the forces acting on the stamp, and the indentation in the stamp.

Let us represent the function $f(x, y)$ in the form

$$f(x, y) = f_+(x, y) + f_-(x, y) \tag{1.5}$$

where $f_+(x, y)$ is a function even in x, and $f_-(x, y)$ is odd in x. Consequently, problem splits into two: 'even' and 'odd' in x.

Below we shall consider only the case even in x; the odd case is completely analogous. Henceforth we shall omit the + symbol from the function $f_+(x, y)$.

Let us seek the solution of (1.1) under the conditions (1.2) and (1.3) as

$$u = \frac{2}{\pi}\int_0^\infty U(\alpha, y, z)\sin\alpha x\, d\alpha, \qquad v = \frac{2}{\pi}\int_0^\infty V(\alpha, y, z)\cos\alpha x\, d\alpha$$
$$w = \frac{2}{\pi}\int_0^\infty W(\alpha, y, z)\cos\alpha x\, d\alpha \tag{1.6}$$

Let us substitute (1.6) into (1.1) and let us perform all the differential operations under the integral sign; equating the integrands to zero we obtain the system

$$D^2U - \frac{\alpha}{1-2\nu}\Theta - \frac{2(1-\nu)}{1-2\nu}\alpha^2 U = 0 \qquad \left(D^2 = \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$$
$$(1-2\nu) D^2V + \Theta_y' - \alpha U_y' - (1-2\nu)\alpha^2 V = 0 \tag{1.7}$$
$$(1-2\nu) D^2W + \Theta_z' - \alpha U_z' - (1-2\nu)\alpha^2 W = 0 \qquad (\Theta = V_y' + W_z')$$

Analogously, from (1.2) and (1.3) we obtain the following boundary conditions

for $z = h$

$$U_z' - \alpha W = 0 \quad (-\infty < y < \infty), \qquad V_z' + W_y' = 0 \quad (-\infty < y < \infty)$$
$$(1-2\nu) W_z' + \nu\Theta - \alpha\nu U = 0 \quad (|y| > a), \qquad W = -F(\alpha, y) \quad (|y| \leqslant a) \tag{1.8}$$

for $z = 0$

$$U_z' - \alpha W = 0 \quad (-\infty < y < \infty), \quad V_z' + W_y' = 0 \quad (-\infty < y < \infty) \tag{1.9}$$
$$W = 0 \quad (-\infty < y < \infty)$$

The functions U, V and W decrease as $|y| \to \infty$.

Here $F(\alpha, y)$ is the Fourier cosine transform of the function $f(x, y)$, i.e.

$$f(x, y) = \frac{2}{\pi}\int_0^\infty F(\alpha, y)\cos\alpha x d\alpha, \qquad F(\alpha, y) = \int_0^\infty f(x, y)\cos\alpha x\, dx \qquad (1.10)$$

Furthermore, let us assume that the function $F(\alpha, y)$ satisfies the following conditions:

(1) For any fixed $0 < \alpha < \infty$ it has a continuous first derivative in $y \in [-a, a]$ with the exception of a finite number of points of discontinuity of the first kind;

(2) For any fixed $0 < \alpha < \infty$ it has a finite number of points of discontinuity of the second kind for the second derivative with $y \in [-a, a]$;

(3) For any fixed $0 < \alpha < \infty$ it is strictly monotone in y for $0 < |y| \leqslant a$.

If the function $F(\alpha, y)$ is not strictly monotone in y, then a strictly monotone function $\varphi(\alpha, y)$, may always be selected such that $F_y'(\alpha, y) + \varphi_y'(\alpha, y) > 0$ or $F_y'(\alpha, y) + \varphi_y'(\alpha, y) < 0$, and $F(\alpha, y)$ represented as the combination of two strictly monotone functions.

Let a strictly monotone function $F^*(\alpha, y)$ be a continuation of the function $F(\alpha, y)$ into the interval $a < |y| < \infty$ while retaining all the other properties of the function $F(\alpha, y)$.

Let us make the following change of variables in equations (1.7) and the boundary conditions (1.8) and (1.9) remembering that the problem is even in y

$$\eta = \frac{a - |y|}{h}\omega(|y|), \qquad \zeta = -\frac{z}{h} \qquad (0 \leqslant |y| < \infty,\ 0 \leqslant z \leqslant h) \qquad (1.11)$$

We shall require the following conditions to be fulfilled:

(1) The function η must be strictly monotone and decreasing for $0 < |y| < \infty$ i.e. $\eta'(|y|) < 0$.

(2) The function η has a continuous first derivative in $y \in [-\infty, \infty]$, with the exception of a finite number of points of discontinuity of the first kind, and has a finite number of points of discontinuity of the second kind for the second derivative with $y \in [-\infty, \infty]$.

(3) The function $\eta(y)$ takes on the following values:

$$\eta(a) = 0, \qquad \eta(0) = C_1/\lambda, \qquad \eta(\infty) = -C_2/\lambda$$

Here C_1 and C_2 are positive constants and $C_1 < \infty$, while C_2 may assume infinite value. Let us take the function $\omega(|y|)$ as

$$\omega(|y|) = \frac{F(\alpha, a) - F(\alpha, y)}{F_y'(\alpha, a)(a - |y|)} \qquad \text{for} \quad 0 \leqslant |y| \leqslant a$$
$$\omega(|y|) = \frac{F(\alpha, a) - F^*(\alpha, y)}{F_y^{*\prime}(\alpha, a)(a - |y|)} \qquad \text{for} \quad a \leqslant |y| < \infty \tag{1.12}$$

Then it is easy to show on the basis of the properties of the functions $F(\alpha, y)$ and $F^*(\alpha, y)$, that $\eta = \eta(|y|)$ satisfies all the properties listed above. At the same time the function

$$F(\alpha, y) = b_0(\alpha) + b_1(\alpha)\eta \tag{1.13}$$

for all $0 \leqslant |y| \leqslant a$, where

$$b_0 = F(\alpha, a), \qquad b_1 = -hF_y'(\alpha, a) \tag{1.14}$$

The back substitution of $|y|$ and z with η and ζ asymptotic for small λ may be uniquely represented as

$$|y| = a - h\eta + \ldots, \qquad z = -h\zeta \tag{1.15}$$

Having made the substitutions (1.11) in (1.7), in the boundary conditions (1.8), (1.9), neglecting terms of order h and h^2 in the obtained relationships and putting $1/\lambda = \infty$ $(\lambda = h/a)$, we obtain the following system of differential equations with the boundary conditions

$$D^2U^* = 0, \quad (1-2\nu)D^2V^* + \Theta_\eta^{*\prime} = 0, \quad (1-2\nu)D^2W^* + \Theta_\zeta^{*\prime} = 0 \tag{1.16}$$

for $\zeta = -1$

$$U_\zeta^{*\prime} = 0 \quad (-\infty < \eta < \infty), \qquad V_\zeta^{*\prime} + W_\eta^{*\prime} = 0 \quad (-\infty < \eta < \infty) \tag{1.17}$$
$$(1-2\nu)W_\eta^{*\prime} + \nu\Theta^* = 0 \; (-\infty < \eta < 0), \; W^* = -[b_0(\alpha) + b_1(\alpha)\eta] \; (0 \leqslant \eta < \infty)$$

for $\zeta = 0$

$$U_\zeta^{*\prime} = 0 \quad (-\infty < \eta < \infty) \quad V_\zeta^{*\prime} + W_\eta^{*\prime} = 0 \quad (-\infty < \eta < \infty), \tag{1.18}$$
$$W^* = 0 \quad (-\infty < \eta < \infty)$$

The functions U^*, V^*, W^* decrease as $|\eta| \to \infty$.

At the same time the function $b_0 + b_1\eta$ is continued analytically with extension of the coordinate η into the domain $\lambda^{-1}\omega(0) \leqslant \eta < \infty$.

Now it is easy to see that the considered problem, when the thickness of the layer is taken into account is split into the following two problems.

1. The determination of the solution of the first of the differential equations (1.16) taking into account only the first of the boundary conditions (1.17) and (1.18) and the condition that the function U^* decreases at infinity; it is known, that the solution of this problem is identically equal to zero.

2. The solution of the system consisting of the second and third differential equations of (1.16) taking into account the remaining boundary conditions of (1.17) and (1.18) and the conditions that the functions V^* and W^* decrease at infinity.

The last problem is a contact plane problem on the effect of a semi-infinite flat inclined stamp on an elastic strip of unit thickness. This problem may be reduced [2] by the methods of operational calculus to the solution of the following integral equation in terms of the distribution function of contact pressures $Q^*(\alpha, \tau)$:

$$\int_0^\infty Q^*(\alpha, \tau) K(\tau - \eta)\, d\tau = \pi\chi\,[b_0(\alpha) + b_1(\alpha)\,\eta] \qquad (0 \leqslant \eta < \infty) \qquad (1.19)$$

$$K(\tau - \eta) = \int_0^\infty \frac{L(u)}{u}\cos(\tau - \eta)\,u\,du, \quad L(u) = \frac{\cosh 2u - 1}{\sinh 2u + 2u}, \qquad \chi = \frac{G}{1 - \nu} \qquad (1.20)$$

Evidently, the function $Q^*(\alpha, \eta)$, is connected with $q(x, y)$ by means of the relationship*

$$q(x, y) \approx \frac{2}{\pi h}\int_0^\infty Q^*(\alpha, \eta)\cos\alpha x\, d\alpha = \frac{2}{\pi h}\int_0^\infty Q(\alpha, y)\cos\alpha x\, d\alpha \qquad (1.21)$$

Thus, the asymptotic solution of the considered problem is for small values of the parameter λ, determined by (1.21) if the solution of (1.19) is known. The connection between the stress acting in each cross-section of the stamp and its settling is determined from the formula

$$P(x) = \int_{-a}^{a} q(x, y)\, dy \qquad (1.22)$$

The solution of the problem vanishing at $y = \pm a$ will when a is fixed evidently exist if

$$\lim_{y \to \pm a} \sqrt{a^2 - y^2}\int_0^\infty Q(\alpha, y)\cos\alpha x\, d\alpha = 0 \qquad (1.23)$$

This relationship imposes definite constraints on the function $f(x, y)$ which is of the form (1.10).**

If the function $f(x, y)$ is periodic with the period of $2l$, then the Fourier integral (1.10) transforms into the Fourier series

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* Formula (1.21) was obtained by an appropriate transformation of the Huhn's law formula for σ_z.

** All the presented results may be obtained by means of an asymptotic solution of the integral equation of the considered contact problem (see (1.29)) for small λ as has been done in [1], say; i.e. its asymptotic solution for small λ is given by formula (1.21).

$$f(x, y) = \sum_{n=0}^{\infty} f_n(y) \cos \frac{\pi n x}{l} \tag{1.24}$$

and the asymptotic solution of the problem for small λ (1.21) evidently becomes

$$q(x, y) = \frac{1}{h} \sum_{n=0}^{\infty} q_n(y) \cos \frac{\pi n x}{l} \tag{1.25}$$

If the function $f(x, y)$ is degenerate, i.e.

$$f(x, y) = \sum_{n=0}^{N} f_n(y) \psi_n(x) \tag{1.26}$$

then it can be shown that the solution of the problem will also be degenerate and representable as

$$q(x, y) = \frac{1}{h} \sum_{n=0}^{N} q_n(y) \psi_n(x) \tag{1.27}$$

where N is any natural number, or infinity.

In (1.25) and (1.27) the function $q_n(y) = q_n(\eta_n)$, where $q_n(\eta_n)$ is the solution of the integral equation (1.19) with the right hand side

$$b_{0n} + b_{1n}\eta_n = f_n(y) \tag{1.28}$$

and where η is constructed by means of (1.11) and (1.12) while the constants b_{0n} and b_{1n} are determined from (1.4).*

It was mentioned above that the function $f(x, y)$ is assumed even in y, however, the solution may be obtained even for the odd function $f(x, y)$ which is in y by utilizing the example given below.

It is known (see [3], say), that the solution of the system of Lamé equations (1.1) which the boundary conditions (1.2) and (1.3) can be reduced by the methods of operational calculus to determination of the contact stresses $q(x, y)$ from the integral equation

$$\begin{gathered} \int_{-a}^{a} \int_{-\infty}^{\infty} q(s, t) K(R/h)\, ds\, dt = 2\pi h \chi f(x, y) \qquad \begin{pmatrix} |y| \leqslant a \\ |x| < \infty \end{pmatrix} \\ K\left(\frac{R}{h}\right) = \int_{0}^{\infty} L(u) J_0(uR/h)\, du, \qquad R = \sqrt{(s-x)^2 + (t-y)^2} \end{gathered} \tag{1.29}$$

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* It is assumed that the functions $f_n(y)$ are strictly monotone in y for all $0 < |y| \leqslant a$ and for any natural n, and that they also satisfy other conditions formulated above for the function $F(\alpha, y)$. If some of the $f_n(y)$ are not strictly monotone, it may always be represented as a combination of two strictly monotone functions.

Let us suppose that we require to find the solution of the integral equation

$$\int_{-a}^{a}\int_{-\infty}^{\infty} p(s,\,t)\,K\left(\frac{R}{h}\right) ds\,dt = 2\pi h\chi g\,(x,\,y)\begin{pmatrix}|\,y\,|\leqslant a\\ |\,x\,|<\infty\end{pmatrix} \tag{1.30}$$

where $f(x, y)$ is an odd function of y. Moreover, let

$$g^*\,(x,\,y)+c\,(x) \tag{1.31}$$

be the primitive of the function $g\,(x, y)$ in y.

Let us find the solution of (1.29) when $f(x, y)$ is equal to (1.31). This solution must vanish for $y = \pm a$, and this will yield a condition for the unique selection of $c\,(x)$. Let us differentiate both sides of (1.29) with respect to y, where $f(x, y)$ has the form (1.31). Afterwards, taking into account that

$$K_y'\,(R\,/\,h) = -\,K_t'\,(R\,/\,h) \tag{1.32}$$

let us transfer the derivative from the kernel to the function $q\,(s, t)$ by integration by parts. We now see without difficulty that the solution of (1.30) is of the form

$$p\,(s,\,t) = q_t'\,(s,\,t) \tag{1.33}$$

where $q\,(s, t)$ is the solution of the first integral equation (1.29) with $f(x, y)$ equal to $g^*\,(x,\,y)+c\,(x)$, and, which vanishes for $y = \pm a$.*

The relation between the moment present at every cross-section of the stamp and its settling is determined by the formula

$$M\,(x) = \int_{-a}^{a} q\,(x,\,y)\,y\,dy \tag{1.34}$$

Let us note that the force $P\,(x)$ and the moment $M\,(x)$ acting at each section of the stamp may also be determined by means of the relationships [2]

$$P\,(x) = \int_{-a}^{a} q_0\,(y)\,f\,(x,\,y)\,dy, \qquad M\,(x) = \int_{-a}^{a} p_1\,(y)\,f\,(x,\,y)\,dy \tag{1.35}$$

where $q_0\,(y)$ and $p_1\,(y)$ are, respectively, the solutions for the case $f\,(x,\,y)\equiv 1$ (flat stamp) and $g\,(x,\,y)\equiv y$ (inclined stamp).

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* If the asymptotic solution of the first integral equation (1.29) is known for small values of the parameter λ, then by (1.33) we shall obviously obtain also the asymptotic solution of (1.30).

2. Solution of the integral equation (1.19). Let us consider a more general integral equation

$$\int_0^\infty Q_n(\tau) K(\tau - \eta)\, d\tau = \pi\chi\eta^n \qquad (0 \leqslant \eta < \infty) \tag{2.1}$$

The closed solution of this equation may evidently be found directly by the Wiener-Hopf method, as it was done in [4]. However, it should be noted that a more direct way exists for the determination of the solution of the integral equation (2.1) for any n, which follows.

Let the solution of the auxiliary integral equation

$$\int_0^\infty Q(\varepsilon, \tau) K(\tau - \eta)\, d\tau = \pi\chi e^{i\eta\varepsilon} \qquad (0 \leqslant \eta < \infty) \tag{2.2}$$

be known.

Differentiating this solution n times with respect to ε and subsequently putting $\varepsilon = 0$ we shall evidently after dividing the result by i^n, obtain the solution of (2.1).

The solution of the integral equations (2.1) may also be obtained by still another method, which is somewhat more complicated than the preceding, but is of considerable theoretical interest. Namely, let us put $\varepsilon = 0$, in the solution of (2.2); then evidently we obtain the solution $Q_0(\tau)$ of (2.1) for $n = 0$.

Let us introduce the function

$$\int_0^\tau Q_0(t)\, dt = Q_1^\circ(\tau) \tag{2.3}$$

Then, let us put $Q_0(\tau) = Q_1^{\circ\prime}(\tau)$ in (2.1) for $n = 0$ and let us transfer the derivative into $(\tau - \eta)$ by integration by parts. Afterwards, noting that $K_\tau'(\tau - \eta) = -K_\eta'(\tau - \eta)$, and integrating the obtained relationship with respect to η between the limits 0 and η, we shall obtain the following integral equation for determination of $Q_1^\circ(\tau)$:

$$\int_0^\infty Q_1^\circ(\tau) K(\tau - \eta)\, d\tau = \pi\chi(\eta + c) \qquad \left(c = \int_0^\infty Q_1^\circ(\tau) K(\tau)\, d\tau\right) \tag{2.4}$$

The term outside the integral, which is obtained in transferring the derivative vanishes since the function $K(t)$ decreases by exponential* as $|t| \to \infty$ while $Q_1^\circ(0) = 0$. It obviously follows from (2.4) that the solution of (2.1) for $n = 1$ is

$$Q_1(\tau) = Q_1^\circ(\tau) - cQ_0(\tau) \tag{2.5}$$

Now, we shall determine the constant c in (2.5); to do this, let us consider the auxiliary integral equation

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* The exponential decrease of the kernel $K(t)$ at infinity, as well as some other properties, will be shown below.

$$\int_{-\infty}^{\infty} v(\tau) K(\tau - \eta) d\tau = \pi \chi \eta \qquad (-\infty < \eta < \infty) \tag{2.6}$$

together with the integral equation (2.1) for $n = 1$.

It is easy to show that the solution of the integral equation (2.1) with $n = 1$ and $T \to \infty$ tends to the solution of the integral equation (2.6), i.e.

$$\lim [Q_1(\tau) - v(\tau)] = 0 \quad \text{for} \quad \tau \to \infty \tag{2.7}$$

This condition may just be used to determine the form of the constant c. The solution of (2.6) is easily found by applying the Fourier transform, and is

$$v(\eta) = \frac{\chi}{\pi} \int_{-\infty}^{\infty} \tau K^*(\tau - \eta) d\tau \qquad \left(K^*(t) = \int_0^{\infty} \frac{u}{L(u)} \cos tu du\right) \tag{2.8}$$

Repeating the above-mentioned system of procedures m times, we obtain the solution of the integral equation (2.1) for $n = m$.

Now, let us determine the solution of (2.2). In order to obtain a solution which is practical, it is necessary to approximate the kernel $K(t)$ by a simpler expression. To do this, let us consider the properties of the functions $L(u)$ and $K(t)$ in more detail.

It is easily seen that the function $L(u)$ has the following properties:

$$L(u) \to Au + O(u^4) \quad \text{for} \quad u \to 0, \quad (A = 1/2); \qquad L(u) \to 1 + O(e^{-2u}) \quad \text{for} \quad u \to \infty \tag{2.9}$$

and it can also be shown that [2]

$$K(t) \sim -\ln|t| + B \quad \text{for} \quad t \to 0 \qquad (B = \text{const}) \tag{2.10}$$

Moreover let us show that for large t the function $K(t)$ decreases exponentially. To do this, we shall consider the auxiliary integral

$$J(t) = \int_\Gamma \frac{L(z)}{z} e^{itz} dz \tag{2.11}$$

where $z = u + iv$, the function $L(z)$ is given by the second relation of (1.20), the contour of integration Γ passes along the real axis* and the semi-circle of radius R in the upper half-plane. Let us represent the function $L(z)/z$ as the ratio of two even functions entire in z, namely

$$L(z) \mid z = P(z) \mid Q(z) \tag{2.12}$$

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* From the second relationship (1.20) it is easily seen that the function $L(z)$ has no poles on the real axis.

Now, letting R tend to infinity in (2.11), taking account of the second property of (2.9) and using the theory of residues, we obtain

$$K(t) = \lim_{R\to\infty} J(t) = \pi i \sum_{k=1}^{\infty} \frac{P(\zeta_k)}{Q'(\zeta_k)} \exp(it\zeta_k) \tag{2.13}$$

Here ζ_k are the roots of the entire function $Q(z)$; and $Q(z)$ has no multiple roots. It follows from (2.13) that

$$K(t) \sim e^{-|t|\varkappa} \quad \text{for} \quad |t| \to \infty \qquad (\varkappa = \inf |\operatorname{Im} \zeta_k|) \tag{2.14}$$

It can also be shown that for $0 < |t| < \infty$ the kernel $K(t)$ is a function, which is continuous and continuously differentiable any number of times.

Now, let us approximate the function $L(u)$, in agreement with (2.9), by the expression

$$L(u) \approx u \frac{\sqrt{u^2 + D^2}}{u^2 + E} \qquad \left(\frac{D}{E} = A\right) \tag{2.15}$$

As can be shown here, that all the fundamental properties of the function $K(t)$ mentioned above are still satisfied. In the problem under consideration, the error in the approximation (2.15) does not exceed 12% for $D = 1$.

Omitting the explanation of the application of the Wiener-Hopf method to the integral equation (2.2) when the approximation (2.15) is used, let us give the final result

$$Q(\varepsilon, \eta) = \frac{e^{i\varepsilon\eta}}{K(\varepsilon)} \operatorname{erf} \sqrt{(D + i\varepsilon)\eta} + \frac{\sqrt{E} - i\varepsilon}{\sqrt{D - i\varepsilon}} \frac{e^{-D\eta}}{\sqrt{\pi\eta}} \left(K(\varepsilon) = \frac{(\varepsilon^2 + D^2)^{1/2}}{(\varepsilon^2 + E)}\right) \tag{2.16}$$

From (2.16), as shown above, we obtain the solution of (2.1) for $n = 0,1$. Analogous solutions may be obtained by utilising the other example given above; it should only be mentioned here that the solution of (2.6) for the considered problem is

$$v(\eta) = \chi\eta / A \tag{2.17}$$

In conclusion, we shall quote the solution of (1.19), which is

$$\begin{gathered} Q^*(\alpha, \eta) = \frac{\chi}{A} \Big\{ [b_0(\alpha) + b_1(\alpha)\eta] \operatorname{erf} \sqrt{D\eta} + \\ + \left[b_0(\alpha) + b_1(\alpha) \left(\frac{\eta}{\sqrt{AD}} + \frac{1}{2D} - \frac{\sqrt{A}}{\sqrt{D}} \right) \right] \frac{\sqrt{A}}{\sqrt{\pi\eta}} e^{-D\eta} \Big\} \end{gathered} \tag{2.18}$$

3. Solution of the contact problem under consideration. Substituting (2.18) into (1.21) and utilizing relationships (1.11), (1.12) and (1.14) we obtain the asymptotic solution of the considered problem for small λ in form

$$\begin{gathered} q(x, y) = \frac{2\chi}{\pi h A} \int_0^\infty \Big\{ F(\alpha, y) \operatorname{erf} \left(D \, \frac{F(\alpha, a) - F(\alpha, y)}{hF_y'(\alpha, a)} \right)^{1/2} + \\ + \left[\left(\sqrt{A} - \frac{1}{D} \right) F(\alpha, a) + \frac{1}{\sqrt{D}} F(\alpha, y) + \left(\frac{A}{\sqrt{D}} - \frac{\sqrt{A}}{2D} \right) hF_y'(\alpha, a) \right] \times \\ \times \frac{1}{\sqrt{\pi}} \left(\frac{F(\alpha, a) - F(\alpha, y)}{hF_y'(\alpha, a)} \right)^{-1/2} \exp \left(-D \frac{F(\alpha, a) - F(\alpha, y)}{hF_y'(\alpha, a)} \right) \Big\} \cos \alpha x \, d\alpha \end{gathered} \tag{3.1}$$

Solutions of the type (1.25) and (1.27) may be represented analogously. The condition (1.23) that the solution (3.1) vanish for $y = \pm a$ is evidently

$$F(\alpha, a) + \left(\frac{\sqrt{A}}{\sqrt{D}} - \frac{1}{2D}\right) hF_y'(\alpha, a) = 0 \tag{3.2}$$

Substituting (3.2) into (3.1), we shall obtain the solution of the considered problem vanishing for $y = \pm a$, as

$$q(x, y) = \frac{2\chi}{\pi h A} \int_0^\infty \left\{F(\alpha, y) \operatorname{erf}\left(D \frac{F(\alpha, a) - F(\alpha, y)}{hF_y'(\alpha, a)}\right)^{1/2} - \right.$$
$$\left. - \left(\frac{hF_y'(\alpha, a)}{\pi D} [F(\alpha, a) - F(\alpha, y)]\right)^{1/2} \exp\left(-D \frac{F(\alpha, a) - F(\alpha, y)}{hF_y'(\alpha, a)}\right)\right\} \cos \alpha x \, d\alpha \tag{3.3}$$

Now, let $f(x, y) = g^*(x, y) + c(x)$. Let us determine the function $c(x)$ from the condition (3.2) that the solution becomes zero on the edges of the stamp

$$c(\alpha) = -G^*(\alpha, a) - \left(\frac{\sqrt{A}}{\sqrt{D}} - \frac{1}{2D}\right) hG_y^{*\prime}(\alpha, a) \tag{3.4}$$

Here $c(\alpha)$ and $G^*(\alpha, y)$ are cosine Fourier transforms of the functions $c(x)$ and $g^*(x, y)$. Substituting the transform $F(\alpha, y) = G^*(\alpha, y) + c(\alpha)$ into (3.3) and then differentiating both sides with respect to y, we shall obtain, in conformity with (1.33), the asymptotic solution for small λ of our problem for the case of the function $g(x, y)$ odd in y, of the displacement of the surface of the elastic layer under the stamp:

$$p(x, y) = \frac{2\chi}{\pi h A} \int_0^\infty G(\alpha, y) \left[\operatorname{erf}\left(D \frac{G^*(\alpha, a) - G^*(\alpha, y)}{hG_y^{*\prime}(\alpha, a)}\right)^{1/2} + \right.$$
$$\left. + \left(\pi \frac{G^*(\alpha, a) - G^*(\alpha, y)}{AhG_y^{*\prime}(\alpha, a)}\right)^{-1/2} \exp\left(-D \frac{G^*(\alpha, a) - G^*(\alpha, y)}{hG_y^{*\prime}(\alpha, a)}\right)\right] \cos \alpha x \, d\alpha \tag{3.5}$$

where $G(\alpha, y)$ is the cosine Fourier transform of the function $g(x, y)$.

Let us now consider the case when the Fourier cosine-transform $F(\alpha, y)$ of the function $f(x, y)$ is not strictly monotone in y. In this case, as remarked in section 1, it is necessary to represent the function $F(\alpha, y)$ as

$$F(\alpha, y) = \varphi(\alpha, y) - \psi(\alpha, y) \tag{3.6}$$

where the functions $\varphi(\alpha, y)$ and $\psi(\alpha, y)$ are strictly monotone in y and satisfy other properties mentioned in section 1. Then the asymptotic solution of our problem may be presented for small λ, as the combination

$$q(x, y) = q_1(x, y) - q_2(x, y) \tag{3.7}$$

where $q_1(x, y)$ and $q_2(x, y)$ are determined from (3.1) in which $F(\alpha, y)$ is equal to $\psi(\alpha, y)$ and $\varphi(\alpha, y)$ respectively.

For the particular case of $f(x, y) \equiv f(x)$, which as will be shown later plays an important part, the solution (3.7) becomes

$$q(x, y) = \frac{2\chi}{\pi h A} \int_0^\infty F(\alpha) \left[\operatorname{erf} \left(D \frac{\varphi(\alpha, a) - \varphi(\alpha, y)}{h\varphi_y'(\alpha, a)} \right)^{1/2} + \right. \tag{3.8}$$

$$\left. + \left(\pi \frac{\varphi(\alpha, a) - \varphi(\alpha, y)}{Ah\varphi_y'(\alpha, a)} \right)^{-1/2} \exp \left(-D \frac{\varphi(\alpha, a) - \varphi(\alpha, y)}{h\varphi_y'(\alpha, a)} \right) \right] \cos \alpha x \, d\alpha$$

where $F(\alpha)$ is the cosine Fourier transform of the function $f(x)$.

The force and the moment acting in a section of the stamp may be determined by formulas (1.22), (1.34) or (1.35). After simple manipulations the formula (1.35) for the force may be represented as

$$P(x) = \frac{2}{\pi} \int_0^\infty \cos \alpha x \, d\alpha \int_{-a}^{a} F(\alpha, y) \, Q_0(\alpha, y) \, dy \tag{3.9}$$

where $Q_0(\alpha, y)$ is an expression contained in square brackets in (3.8), in which, not unexpectedly, $\varphi(\alpha, y)$ is replaced by the function $F(\alpha, y)$.

It is easy to see from (3.7) and (3.8) that the asymptotic solution of the considered problem for small λ is not determined uniquely, namely, some arbitrariness exists in the selection of the function $\varphi(\alpha, y)$, which possesses only the properties mentioned in section 1. Evidently, the solutions obtained are asymptotically equivalent for small λ, however, the range of their practical utilization in λ apparently depends on the success in selecting the function $\varphi(\alpha, y)$. The exact limits of the practical utilization of the obtained solution may be learned only by constructing the next term of the asymptotic solution of the considered problem for small values of λ. Here this question will not be considered; however, specific computations have shown that all the obtained solutions may be used confidently at least in the range $0 < \lambda \leqslant 1/2$.

Here, we shall select the function $\varphi(\alpha, y)$ by its suitability. Namely: (1) if the function $F(\alpha, y)$ is not strictly monotone in $|y|$, then such function $\varphi(\alpha, y)$ should be chosen, which is analytic in y for $0 \leqslant |y| \leqslant a$, and which does not carry its asymptotically small singularities for small λ into the solution of the considered problem, (2) if the function $F(\alpha, y)$ is strictly monotone on $0 \leqslant |y| \leqslant a$ then the function $\varphi(y, \alpha)$ identically equal to zero should be chosen, (3) if the function $F(\alpha, y) \equiv F(\alpha)$, which corresponds to a stamp plane in y, then the function $\varphi(\alpha, y)$ ib (3.8) is arbitrary, however, if $F(\alpha)$ is combined with the function* $G(\alpha, y)$, odd in y, then such function $\varphi(\alpha, y)$ should be chosen, in (3.8) which is equal to $G^*(\alpha, y)$.

Let us consider the plane problem, namely the case when $f(x, y) \equiv f(y)$. Evidently all the formulas obtained earlier may be used here if we remember that the cosine Fourier transform for the function $f(x, y)$ becomes

$$F(\alpha, y) = \pi \delta(\alpha) f(y) \tag{3.10}$$

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* An analogous fact holds for the general case of (3.1).

where δ (α) is the Dirac delta function, and if we utilize the known properties of the function δ (α). For example, (3.1) may be represented as

$$q(y)=\frac{\chi}{hA}\left\{f(y)\operatorname{erf}\left(D\,\frac{f(a)-f(y)}{hf'(a)}\right)^{1/2}+\left[\left(\sqrt{A}-\frac{1}{\sqrt{D}}\right)f(a)+\frac{1}{\sqrt{D}}f(y)+\right.\right.$$
$$\left.\left.+\left(\frac{A}{\sqrt{D}}-\frac{\sqrt{A}}{2D}\right)hf'(a)\right]\frac{1}{\sqrt{\pi}}\left(\frac{f(a)-f(y)}{hf'(a)}\right)^{-1/2}\exp\left(-D\,\frac{f(a)-f(y)}{hf'(a)}\right)\right\} \tag{3.11}$$

The remaining formulas (3.3), (3.5) and (3.8) assume an analogous form.

It is easy to note, that the solution (3.1) within the line of contact $[-a, a]$ tends rapidly to the corresponding degenerate solution defined by the formula*

$$q\ (y)\ =\ \chi f\ (y)\ /\ Ah,\qquad |y|\leqslant a \tag{3.12}$$

If the function $f(y)$ has angular points $y = b_i$ within the line of contact, then from (3.11) and (3.12) it follows that the solution of the problem $q(y)$ will behave as $|y - b_i|$ at these points.

If the function $g(y)$ has discontinuities of the first kind at the points $y = c_i$ within the line of contact, then it may similarly be established that the solution of the problem $p(y)$ will behave as sign $(y - c_i)$ at these points.

However, it is known [5], that in the case of an elastic half-space the solution of the contact problem has logarithmic singularities, at the angular points of the function $f(y)$ and removable poles at the points of discontinuity of the first kind of the function $g(y)$. It can be proved that such singularities are also retained in the considered case of the contact problem for a layer. Hence, we find that the obtained solutions misbehave at the angular points of the function $f(y)$ and at the points of discontinuity of the first kind of the function $g(y)$. This however, has no practical effect on the accuracy of the solution of the considered problem, for sufficiently small λ on the intervals between the mentioned points on the segment $y \in [-a, a]$, and on the integral characteristics of the solution, i.e. the force and the moment.

The asymptotic solution may be obtained for a small relative thickness of the layer λ which has the necessary singularities at the above-mentioned points. A more accurate solution can be given in the form

$$q(y)=q_1(y)+q_2(y)-\frac{\chi f(y)}{Ah} \tag{3.13}$$

where $q_1(y)$ is given by (3.11) together with the corresponding formula obtained from (3.5), and $q_2(y)$ is

$$q_2(y)=\frac{\chi}{\pi h^2}\int_{-\infty}^{\infty} f(\eta)\,K^*\left(\frac{\eta-y}{h}\right)d\eta \tag{3.14}$$

or, taking into account the approximation (2.15)

* Analysis of the case $F(\alpha, y) = F(\alpha) + G(\alpha, y)$ is utilized here.

$$q_2(y)=\frac{\chi}{\pi h^2}\int_{-\infty}^{\infty}f(\eta)\left[\left(\frac{D}{A}-D^2\right)K_0\left(D\,\frac{\eta-y}{h}\right)-\frac{Dh}{\eta-y}K_1\left(D\,\frac{\eta-y}{h}\right)\right]d\eta \quad (3.15)$$

It may be shown that $q_2(y)$ has logarithmic and removable singularities at appropriate points, and also tends rapidly to the degenerate solution of the form (3.12) on departure from these points.

Let us note that in the case of plane problems $(f(x,y)\equiv f(y))$, the limits of applicability of the asymptotic solutions for small λ obtained in the manner described above, may be broadened by application of the following method of correction.

Let $q_0(y)$ be the asymptotic solution of the plane problem for a flat stamp, which may be determined from (3.8). Evidently, the expression $q^*(y)=q_0(y)\,\varkappa(\lambda)$ is also an asymptotic solution of the same problem if $\varkappa(\lambda)\to 1$ as $\lambda\to 0$, i.e. for small λ it asymptotically satisfies the integral equation

$$\int_{-a}^{a}q^*(\eta)\,d\eta\int_0^{\infty}\frac{L(u)}{u}\cos\left(\frac{\eta-y}{h}u\right)du=\pi\chi \quad (3.16)$$

for all values $|y|\leqslant a$, which are obtained from (1.29) for the plane problem case. In particular, for $y=0$ we obtain the following expression for the correction factor $\varkappa(\lambda)$ from (3.16):

$$\varkappa(\lambda)=\pi\chi\left[\int_{-a}^{a}q_0(\eta)\,d\eta\int_0^{\infty}\frac{L(u)}{u}\cos\left(\frac{\eta u}{h}\right)du\right]^{-1} \quad (3.17)$$

The integral (3.17) may be tabulated in terms of the parameter λ on an electronic computer. Calculations have shown that the more exact solution $q^*(y)$ has wider limits of applicability, namely, it may be used with sufficient accuracy over the range $0<\lambda<2$.

Let us present some results of the calculations. Putting $F(\alpha)=\delta(\alpha)$ in (3.8) and choosing $\varphi(\alpha,y)\equiv y^2$, we obtain the expression for $q_0(y)$ as

$$q_0(y)=\frac{\chi}{A\lambda a}\left\{\operatorname{erf}\left[\frac{D}{2\lambda}\left(1-\frac{y^2}{a^2}\right)\right]^{1/2}+\sqrt{\frac{2A\lambda}{\pi}}\left(1-\frac{y^2}{a^2}\right)^{-1/2}\exp\left[-\frac{D}{2\lambda}\left(1-\frac{y^2}{a^2}\right)\right]\right\} \quad (3.18)$$

Let us also give an expression for the force P obtained by means of (1.22) together with (3.18)

$$P=\frac{\chi}{A\lambda}\left(\frac{\pi D}{2\lambda}\right)^{1/2}e^{-D/4\lambda}\left[\left(1+2\lambda\frac{\sqrt{A}}{\sqrt{D}}\right)(I_0(D/4\lambda)+I_1(D/4\lambda)\right] \quad (3.19)$$

Presented in the table are the computed values of the correction factor $\varkappa(\lambda)$, obtained by means of (3.17) and (3.18), as well as the values of the stresses $q_0(y)$ and

$q^*(y)$ and their corresponding, (practically exact) values of the stresses taken from [4]. The last two columns give the values of $\omega = \lim q(y)\sqrt{a^2-y^2}$ for $y \to a$ and of the force P.

	λ	$\varkappa$	x/a						
			0	0.4	0.6	0.8	0.95	ω/χ	$P/\chi a$
$\frac{aq^0}{\chi}$	1/2		3.95	3.97	4.02	4.27	5.89	1.59	8.91
	1		2.05	2.09	2.18	2.48	3.93	1.13	5.03
	2		1.14	1.19	1.28	1.54	2.67	0.80	3.05
$\frac{aq^*}{\chi}$	1/2	1.00	3.95	3.97	4.02	4.27	5.89	1.59	8.91
	1	0.93	1.90	1.94	2.02	2.30	3.65	1.05	4.67
	2	0.87	0.99	1.04	1.12	1.35	2.33	0.70	2.66
$\frac{aq}{\chi}$	1/2		3.97	3.94	3.90	4.00	5.74	1.58	8.71
	1		1.92	1.93	1.99	2.24	3.80	1.12	4.70
[4]	2		0.97	1.01	1.10	1.38	2.56	0.79	2.75

Having constructed the more accurate solution of the problem $q^*(y)$ for a flat stamp, by means of the Krein [6] formula, we can determine the corresponding solution of the problem for any shape of the stamp. The approximate solutions thus found, together with the corresponding approximate solutions [2 and 7] of the method of large λ, cover the whole range of variation of the parameter $\lambda \in (0,\infty)$ with an accuracy sufficient for practical utilization.

In conclusion, let us consider a specific example $f(x, y) \equiv |y|$ (plane problem). By means of (3.11), we obtain without difficulty, the asymptotic solution for small λ in the form

$$q(y) = \frac{\chi}{\lambda A}\left\{\frac{|y|}{a}\operatorname{erf}\left(D\frac{a-|y|}{h}\right)^{1/2} + \left[\left(\sqrt{A}-\frac{1}{\sqrt{D}}\right)+\frac{1}{\sqrt{D}}\frac{|y|}{a}+\left(\frac{A}{\sqrt{D}}-\frac{\sqrt{A}}{2D}\right)\lambda\right]\times \right.$$
$$\left. \times\frac{1}{\sqrt{\pi}}\left(\frac{a-|y|}{h}\right)^{-1/2}\exp\left(-D\frac{a-|y|}{h}\right)\right\} \tag{3.20}$$

We obtain the expression for the force by means of (1.22) and (1.35)

$$P = \frac{2\chi a}{\lambda A}\left\{\left[\frac{1}{2}+\left(\frac{\sqrt{A}}{\sqrt{D}}+\frac{A}{D}-\frac{\sqrt{A}}{2D\sqrt{D}}-\frac{1}{2D}\right)\lambda-\frac{1}{8}\frac{\lambda^2}{D^2}\right]\operatorname{erf}\sqrt{\frac{D}{\lambda}}+\right.$$
$$\left.+\left(\frac{1}{2}+\frac{5}{2}\frac{\lambda}{D}\right)\left(\frac{\lambda}{D\pi}\right)^{1/2}e^{-D/\lambda}\right\} \tag{3.21}$$

$$P = \frac{2\chi a}{\lambda A}\left\{\left[\frac{1}{2}+\left(\frac{\sqrt{A}}{\sqrt{D}}-\frac{1}{2D}\right)\lambda+\left(\frac{3}{8D^2}-\frac{1}{2}\frac{\sqrt{A}}{\sqrt{D}}\right)\lambda^2\right]\operatorname{erf}\left(\frac{D}{\lambda}\right)^{1/2}+\right.$$

$$\left.+\left[\frac{1}{2}+\left(\frac{3}{2D}+\frac{\sqrt{A}}{\sqrt{D}}\right)\lambda\right]\left(\frac{\lambda}{D\pi}\right)^{1/2}e^{-D/\lambda}\right\} \tag{3.22}$$

Now, by (3.5) we easily find the asymptotic solution of the problem for the case $g(x, y) \equiv \operatorname{sign} y$ as

$$p(y) = \frac{\chi \operatorname{sign} y}{hA}\left[\operatorname{erf}\left(D\frac{a-|y|}{y}\right)^{1/2} + \left(\pi\frac{a-|y|}{Ah}\right)^{-1/2}\exp\left(-D\frac{a-|y|}{h}\right)\right] \tag{3.23}$$

The expression for the moment defined by (1.34) has the form (3.22), and the expression defined by (1.35) is representable as

$$M = \frac{2\chi a}{\lambda A}\left\{\left[\frac{1}{2} + \left(\frac{\sqrt{A}}{\sqrt{D}} - \frac{1}{2D}\right)\lambda\right]\operatorname{erf}\left(\frac{D}{2\lambda}\right)^{1/2} + \left(\frac{\lambda}{2\pi D}\right)^{1/2} e^{-D/2\lambda}\right\} \tag{3.24}$$

Both the formulas for the force (3.21) and (3.22) and the moment (3.22) and (3.24), are asymptotically equal

	$\lambda = 1/2$	$1/4$	$1/8$
$P_{(3.21)}$	$= 1.42$	2.33	4.34
$P_{(3.22)}$	$= 1.34$	2.22	4.21
$M_{(3.24)}$	$= 1.23$	2.21	4.21

and yield in the range $0 < \lambda \leqslant 1/2$ results given above, which practically coincide.

Let us note that the obtained results are completely applicable to the analogous contact problem for a layer when its lower boundary is connected rigidly to a nondeformable base. In this case, only the values of A and D in the approximation (2.15) will change.

4. Formulation of the contact problem for an elastic layer *(case of a stamp of circular cross-section).* Let us now consider the problem of the effect of a circular stamp on an elastic layer of slight thickness.

Let us represent the function $f(r, \varphi)$, defining the settling of points of the surface of the elastic layer under the stamp, as

$$f(r, \varphi) = f_+(r, \varphi) + f_-(r, \varphi) \tag{4.1}$$

Here $f_+(r, \varphi)$ and $f_-(r, \varphi)$ are, respectively, the function even and odd in φ. Below we shall consider only the even case, assuming that the odd case is obtained analogously. Hence, in the following, we shall omit the + sign on the function $f_+(r, \varphi)$

We wish to determine the contact pressure under a stamp

$$q(r, \varphi) = -\sigma_z|_{z=h} \qquad (0 \leqslant r \leqslant a) \tag{4.2}$$

the connection between the stress resultants acting on the stamp, and the degree of penetration of the stamp into the layer.

Assuming that the function $f(r, \varphi)$ admits of a Fourier-Bessel series expansion of the form

$$f(r, \varphi) = \sum_{n=0}^{\infty} f_n(r)\cos n\varphi \tag{4.3}$$

let us seek the solution of the system of Lamé equations in cylindrical coordinates with boundary conditions of the considered problem as

$$u(r,\varphi,z)=\sum_{n=0}^{\infty}u_n(r,z)\cos n\varphi,\qquad v(r,\varphi,z)=\sum_{n=0}^{\infty}v_n(r,z)\sin n\varphi$$
$$w(r,\varphi,z)=\sum_{n=0}^{\infty}w_n(r,z)\cos n\varphi \tag{4.4}$$

Then, for the determination of u_n, v_n, and w_n we shall have

$$\frac{1}{1-2\nu}\frac{\partial\theta_n}{\partial r}+\triangle w_n=0\qquad \frac{1}{1-2\nu}\frac{\partial\theta_n}{\partial r}+\triangle u_n-\frac{u_n}{r^2}-\frac{2n}{r^2}v_n=0$$
$$\frac{1}{1-2\nu}\frac{n\theta_n}{r}+\triangle v_n-\frac{v_n}{r^2}-\frac{2n}{r^2}u_n=0 \tag{4.5}$$
$$\left(\triangle=\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{n^2}{r^2},\quad \theta_n=\frac{\partial w_n}{\partial z}+\frac{\partial u_n}{\partial r}+\frac{u_n}{r}+\frac{n}{r}v_n\right)$$

(where ν is the Poisson coefficient).

The corresponding boundary conditions take the following form:

$$\frac{\partial u_n}{\partial z}+\frac{\partial w_n}{\partial r}=0,\qquad \frac{\partial v_n}{\partial z}-\frac{nw_n}{r}=0\qquad (0\leqslant r<\infty)$$
$$\frac{\partial w_n}{\partial z}+\frac{\nu}{1-2\nu}\theta_n=0\quad (a<r<\infty),\qquad w_n=-f_n(r)\quad (0\leqslant r\leqslant a)$$
$$\text{при } z=0 \tag{4.6}$$
$$\frac{\partial u_n}{\partial z}+\frac{\partial w_n}{\partial r}=0\quad (0\leqslant r<\infty),\qquad \frac{\partial v_n}{\partial z}-\frac{nw_n}{r}=0\quad (0\leqslant r\leqslant\infty)$$
$$w_n=0\quad (0\leqslant r<\infty),\qquad u_n,\ v_n,\ w_n\to 0\quad (r\to\infty)$$

Assuming that the functions $f_n(r)$ satisfy, over the range $0\leqslant r\leqslant a$ the properties (1) to (3) mentioned in section 1, let us change the variables in equations (4.5) and the boundary conditions (4.6)

$$\eta=\frac{a-r}{h}\omega(r),\qquad \zeta=-\frac{z}{h}\qquad \begin{pmatrix}0\leqslant r<\infty\\ 0\leqslant z\leqslant h\end{pmatrix} \tag{4.7}$$

where

$$\omega(r)=\frac{f_n(a)-f_n(r)}{f_n'(a)(a-r)}\quad (0\leqslant r\leqslant a),\qquad \omega(r)=\frac{f_n(a)-f_n^*(r)}{f_n^{*\prime}(a)(a-r)}\quad (a\leqslant r<\infty) \tag{4.8}$$

Here $f_n^*(r)$ is an arbitrary, strictly monotone function, smoothly continuing the function $f_n(r)$ into the domain $a\leqslant r<\infty$.

The reverse change for small λ has the form $r=a-h\eta+\ldots$, $z=-h\zeta$. After the change of variables (4.7) in the above-mentioned equations and boundary conditions, we neglect the terms of order h and h^2 in the obtained relationships, and put

$\lambda^{-1} = \infty$, to obtain the following system of differential equations and boundary conditions

$$D^2 v_n^* = 0 \qquad \left(D^2 = \frac{\partial^2}{\partial\eta^2} + \frac{\partial^2}{\partial\zeta^2}\right)$$
$$(1-2\nu) D^2 u_n^* + \theta_\eta^{*\prime} = 0 \qquad \left(\theta^* = \frac{\partial u_n^*}{\partial\eta} + \frac{\partial w_n^*}{\partial\zeta}\right) \tag{4.9}$$
$$(1-2\nu) D^2 w_n^* + \theta_\zeta^{*\prime} = 0$$

for $\zeta = -1$

$$v_{n\zeta}^{*\prime} = 0 \quad (-\infty < \eta < \infty), \qquad u_{n\zeta}^{*\prime} + w_{n\eta}^{*\prime} = 0 \quad (-\infty < \eta < \infty)$$
$$(1-2\nu) w_{n\eta}^{*\prime} + \nu\theta^* = 0 \quad (-\infty < \eta < 0), \tag{4.10}$$
$$w_n^* = -(b_{n0} + b_{n1}\eta) \quad (0 \leqslant \eta < \infty)$$

for $\zeta = 0$

$$v_{n\zeta}^{*\prime} = 0 \quad (-\infty < \eta < \infty), \qquad u_{n\zeta}^{*\prime} + w_{n\eta}^{*\prime} = 0 \quad (-\infty < \eta < \infty)$$
$$v_n^* = 0 \quad (-\infty < \eta < \infty), \qquad u_n^*,\ v_n^*,\ w_n^* \to 0 \quad (\eta \to \infty)$$

Here

$$b_{n0} = f_n(a), \qquad b_{n1} = -hf_n'(a) \tag{4.11}$$

We see that similarly to section 1, the original problem has split into two problems. The first is determined by the first equation of (4.9) and the first and fifth boundary conditions of (4.10), and its solution is identically equal to zero; the second problem, defined by the second and third equations of (4.9) and the remaining boundary conditions of (4.10), is the plane contact problem on the effect of a semi-infinite, plane, inclined stamp, on an elastic strip of unit thickness. The last problem reduces [2] to the solution of the integral equation

$$\int_0^\infty q_n^*(\tau) K(\tau - \eta)\, d\tau = \pi\chi (b_{n0} + b_{n1}\eta) \qquad (0 \leqslant \eta < \infty) \tag{4.12}$$

where $K(\tau - \eta)$ and χ are given by formulas (1.14).

Having solved the integral equation (4.12), we can find the solution of the original problem $q(r, \varphi)$ from the formula

$$q(r, \varphi) = \sum_{n=0}^\infty q_n(r) \cos n\varphi = \frac{1}{h} \sum_{n=0}^\infty q_n^*(\eta) \cos n\varphi \tag{4.13}$$

The connection between the stresses acting on the stamp and its displacement is determined by the formulas

$$P = 2\pi \int_0^a q_0(r)\, r\, dr, \qquad M_y = \pi \int_0^a q_1(r)\, r^2\, dr \tag{4.14}$$

The solution of the problem which vanishes at $r = a$ for the fixed a does evidently occur under the condition

$$\lim_{r\to a} \sqrt{a^2 - r^2} \sum_{n=0}^{\infty} q_n(r) \cos n\varphi = 0 \tag{4.15}$$

which imposes specific constraints on the function $f(r, \varphi)$.

Let us note that the force P and the moment M_y can also be determined by means of the relationships [8]

$$P = 2\pi \int_0^a p_0(r) f_0(r)\, r\, dr, \qquad M_y = \pi \int_0^a p_1(r) f_1(r)\, r d \tag{4.16}$$

where $p_0(r)$ and $\cos \varphi p_1(r)$ are, respectively, the solutions for the cases $f(r, \varphi) \equiv 1$ (flat stamp), and $f(r, \varphi) = r \cos \varphi$ (inclined stamp).

5. Some representations of the solution of not axially symmetric *contact problems for a stamp of circular cross-section.* We shall show that the solution of the contact problem for a circular stamp with an arbitrary base $f(r, \varphi)$ can always be represented as a combination of the solutions of axisymmetric problems of definite smoothness on the contour $r = a$, each of which is acted upon by some differential operator.

In fact, by assuming the possibility of expanding the function $f(r, \varphi)$ into a Fourier-Bessel series of the form (4.3), and by using known trigonometric formulas, one may come to the conclusion that the method of obtaining a solution for the particular case

$$f(r, \varphi) = \Psi_n(r) \cos^n \varphi \tag{5.1}$$

is sufficient for solution of the general problem.

Let us show that the function $\Psi_n(r) \cos^n \varphi$ may always be represented as

$$\Psi_n(r) \cos^n \varphi = \sum_{k=1}^{n} \Phi_{kx}^{(k)}(r) \tag{5.2}$$

Let us differentiate the arbitrary function $\Phi_n(r)$ n times with respect to x to obtain

$$\Phi_{nx}^{(n)}(r) = \sum_{k=1}^{n} D_k(\Phi_n) \cos^k \varphi \qquad \left(D_n = r^n \left(\frac{1}{r}\frac{d}{dr}\right)^n\right) \tag{5.3}$$

Here D_k is some differential operator with respect to r and of order k. For $k = n$ it is shown above in parentheses.

The equality (5.3) may be rewritten as

$$D_n(\Phi_n)\cos^n\varphi = \Phi_{nx}^{(n)}(r) - \sum_{k=1}^{n-1} D_k(\Phi_n)\cos^k\varphi \tag{5.4}$$

Let us take the function $\Phi_n(r)$ in the form

$$\Phi n(r) = D_n^{-1}(\Psi_n) \tag{5.5}$$

Here the operator D_n^{-1}, is evidently an integral one and determines the function $\Phi_n(r)$ with the accuracy of up to the $(n-1)$-th order polynomial.

Then (5.4) may be rewritten as

$$\Psi_n(r)\cos^n\varphi = \Phi_{nx}{}^{(n)}(r) - \sum_{k=1}^{n-1} D_k[D_n{}^{-1}(\Psi_n)]\cos^k\varphi \tag{5.6}$$

Utilizing the relationship (5.6), which is valid for any n, and using the notation

$$\Psi_{n-1}(r) = D_{n-1}[D_n{}^{-1}(\Psi_n)] \tag{5.7}$$

we obtain

$$\Psi_n(r)\cos^n\varphi = \Phi_{nx}^{(n)}(r) + \Phi_{(n-1)x}^{(n-1)}(r) - \sum_{k=1}^{n-2} D_k[D_n{}^{-1}(\Psi_n) + D_{n-1}{}^{-1}(\Psi_{n-1})]\cos^k\varphi \tag{5.8}$$

where the function $\Phi_{n-1}(r)$ is determined by (2.5) to the accuracy of up to the $(n-2)$-th order polynomial. Continuing this process further, we arrive at (5.2).

Having determined all the functions $\Phi_k(r)$ $(k = 1, 2, \ldots, n)$ with the accuracy of up to the $(k-1)$-th order polynomials, we then find the solution $q_k(r)$ of the axisymmetric contact problems for stamps with bases of the form

$$f(r, \varphi) = \Phi_k(r) \tag{5.9}$$

Afterwards, the coefficients of the arbitrary polynomials are determined from the following linear algebraic systems

$$\lim_{r\to a}\frac{d^s}{dr^s}q_k(r) = 0 \qquad \begin{pmatrix} s = 0, 1, \ldots, k-1 \\ k = 1, 2, \ldots, n \end{pmatrix} \tag{5.10}$$

Let us now show that the solution of the problem corresponding to the case (5.1) may be written as

$$q(r, \varphi) = \sum_{k=0}^{n} q_{kx}^{(k)}(r) \tag{5.11}$$

Indeed, differentiating k times with respect to x the identical equality (see (1.23) for comparison)

$$\iint\limits_{s^2+t^2\leqslant a^2} q_k(\rho)K(R/h)\,ds\,dt = 2\pi h\chi\Phi_k(r) \qquad \begin{pmatrix} r^2 = x^2 + y^2 \\ \rho^2 = s^2 + t^2 \end{pmatrix} \tag{5.12}$$

and integrating the left hand side by parts k times, we obtain

$$\iint\limits_{s^2+t^2\leqslant a^2} q_{ks}^{(k)}(\rho)\, K(R/h)\, ds dt = 2\pi h\chi \Phi_{ks}^{(k)}(r) \tag{5.13}$$

The terms outside the integral disappeared, as a result of (5.10).

The equalities (5.13) and (5.2) confirm the validity of (5.11).

If the parameter λ is small, the algorithm given above which is valid for all $\lambda \in (0, \infty)$, is greatly simplified. Namely, it follows from the results of section 4 that the solution of the problem in the case (5.1) may be written as $q(r,\varphi) = q_n(r)\cos^n\varphi$, where $q_n(r)$ is the solution of the axisymmetric problem for the case $f(r,\varphi) = \psi_n(r)$.

6. Solution of the considered contact problem (circle) As shown in section 4, the solution of the considered contact problem reduces to the determination of the function $q_n^*(\tau)$ from the Wiener-Hopf integral equation (4.12). A solution of this integral equation suitable for the following is given in section 2.

Using (2.21) together with (4.7), (4.8), (4.11) and (4.13), we obtain the asymptotic solution of the problem for small λ

$$q(r,\varphi) = \frac{\chi}{hA}\sum_{n=0}^{\infty}\Big\{f_n(r)\,\mathrm{erf}\Big(\frac{f_n(a)-f_n(r)}{hf_{nr}'(a)}\Big)^{1/2} + \Big[\Big(\sqrt{A}-\frac{1}{\sqrt{D}}\Big)f_n(a) + \frac{1}{\sqrt{D}}f_n(r) + \\ + \Big(\frac{A}{\sqrt{D}} - \frac{\sqrt{A}}{2D}\Big)hf_{nr}'(a)\Big]\frac{1}{\sqrt{\pi}}\Big(\frac{f_n(a)-f_n(r)}{hf_{nr}'(a)}\Big)^{-1/2}\exp\Big(-D\,\frac{f_n(a)-f_n(r)}{hf_{nr}'(a)}\Big)\Big\}\cos n\varphi \tag{6.1}$$

Formula (6.1) is entirely analogous to (3.1). On the basis of (6.1) the solution of the considered problem, also analogous to (3.2) and (3.3), which vanishes for $r = a$ may be obtained. If the functions $f_n(r)$, or some of them, are not strictly monotone in $0 < r \leqslant a$, then it is again necessary to repeat all the considerations which are presented for this case in section 1.

Let us note that (6.1) and other formulas based on, if may, as well as the formulas of section 3 be used confidently, at least in the range

$$0 < \lambda = h/a \leqslant 1/2$$

The case of the axisymmetric problem which plays a large part, as was shown in in section 5, is obtained under the conditions $f_n(r) \equiv 0$ $(n = 1, 2, \ldots)$. For this case we can in the manner analogous to that in section 3 for the case of the plane problem, construct additional boundary layers at the angular points of the functions $f_n(r)$ which, in combination with the fundamental solution of the problem (6.1), afford a possibility of obtaining the requisite singularities at these points.

On the basis of (6.1) we shall obtain the asymptotic solution for small λ of the considered contact problem for the case $f(r) \equiv 1$ (plane stamp) as

$$q(r) = \frac{\chi}{hA}\Big[\mathrm{erf}\Big(D\,\frac{\varphi(a)-\varphi(r)}{h\varphi_r'(a)}\Big)^{1/2} + \Big(\pi\,\frac{\varphi(a)-\varphi(r)}{h\varphi_r'(a)}\Big)^{-1/2}\exp\Big(-D\,\frac{\varphi(a)-\varphi(r)}{h\varphi_r'(a)}\Big)\Big] \tag{6.2}$$

where $\varphi(r)$ is an arbitrary function, strictly monotone in $0 < r \leqslant a$ which also satisfies other properties mentioned in section 1.

The limits of applicability of (6.2) may be extended to $\lambda = 2$ by the insertion of a correction factor just as it was done in section 3. After this, more accurate solutions for any non-plane stamp may in the case of axial symmetry, be determined by means of Krein [6] formula, which in combination with the corresponding solutions of the method of large λ [3 and 8 to 10] will cover the whole range of variation of the parameter $\lambda \in (0, \infty)$ with sufficient accuracy. Taking into account the formulas given in section 5, which connect the non-axisymmetric and the axisymmetric contact problems for a circular stamp, we can conclude that the whole range of variation of λ can be covered by simple formulas for the effect of a circular stamp with an arbitrary base, with accuracy sufficient for practical utilization.

Let us now present some specific examples.

Let us consider the case of a parabolic stamp $f(r, \varphi) \equiv r^2$. Utilizing (6.1) we obtain the solution of the problem in the form

$$q(r) = \frac{\chi a^2}{hA}\left\{\frac{r^2}{a^2}\operatorname{erf}\left[\frac{D}{2\lambda}\left(1-\frac{r^2}{a^2}\right)\right]^{1/2} + \left[\left(\sqrt{A}-\frac{1}{\sqrt{D}}\right)+\frac{1}{\sqrt{D}}\frac{r^2}{a^2}+\right.\right.$$
$$\left.\left.+\left(\frac{A}{\sqrt{D}}-\frac{\sqrt{A}}{2D}\right)\lambda\right]\frac{1}{\sqrt{\pi}}\left[\frac{1}{2\lambda}\left(1-\frac{r^2}{a^2}\right)\right]^{-1/2}\exp\left[-\frac{D}{2\lambda}\left(1-\frac{r^2}{a^2}\right)\right]\right\} \qquad (6.3)$$

The expression for the force obtained by means of (4.14) is

$$P = \frac{\pi\chi a^3}{2A\lambda}\left\{[1 + (2\sqrt{AD}-1)\,2p^{-2} + (4AD - 2\sqrt{AD} - 1)\,p^{-4}]\operatorname{erf}\frac{p}{\sqrt{2}} - \right.$$
$$\left. - 2(1-p^2)\frac{\exp(-{}^1/_2 p^2)}{p\sqrt{2\pi}}\right\} \qquad \left(p = \left(\frac{D}{\lambda}\right)^{1/2}\right) \qquad (6.4)$$

In the case of a conical stamp ($f(r, \varphi) \equiv r$) we find by (6.1)

$$q(r) = \frac{\chi}{\lambda A}\left\{\frac{r}{a}\operatorname{erf}\left(D\frac{a-r}{h}\right)^{1/2} + \left[\left(\sqrt{A}-\frac{1}{\sqrt{D}}\right)+\frac{1}{\sqrt{D}}\frac{r}{a}+\left(\frac{A}{\sqrt{D}}-\frac{\sqrt{A}}{2D}\right)\lambda\right]\times\right.$$
$$\left.\times\frac{1}{\sqrt{\pi}}\left(\frac{a-r}{h}\right)^{-1/2}\exp\left(-D\frac{a-r}{h}\right)\right\} \qquad (6.5)$$

The corresponding value of the force obtained by using (4.14) is given by

$$P = \frac{2\pi\chi a^2}{A\lambda}\left\{\left[\frac{1}{3}+\left(\sqrt{AD}-\frac{1}{2}\right)p^{-2}+\left(AD-\sqrt{AD}+\frac{3}{4}+\frac{1}{2}\sqrt{D}\right)p^{-4}+\right.\right. \qquad (6.6)$$
$$\left.+\left(\frac{1}{4}\sqrt{AD}+\frac{3}{4}\sqrt{D}-\frac{1}{2}AD-\frac{5}{8}\right)p^{-6}\right]\operatorname{erf}p+\left[\frac{7}{6}+(\sqrt{AD}-\frac{3}{2}-2\sqrt{D})\,p^{-2}+\right.$$
$$\left.\left.+\left(\frac{5}{4}+AD-\frac{1}{2}\sqrt{AD}-\frac{3}{2}\sqrt{D}\right)p^{-4}\right]\frac{\exp(-p^2)}{p\sqrt{\pi}}\right\}$$

Now, let us consider the case of the penetration of a flat inclined stamp $f(r, \varphi) = r\cos\varphi$.

The asymptotic solution of this problem for small λ may be obtained in two ways.

Firstly, we may use (4.13) and write the solution in form of (6.5) multiplied by $\cos\varphi$.

Secondly, the solution of this problem is obtained if the solution for a stamp with the base

$$f(r,\varphi) = r^2 + c \tag{6.7}$$

determined from (6.1) is differentiated with respect to x. Here c is selected by the method given in section 5 from the condition of the solution for (6.7) vanishing at $r = a$.

Performing all the mentioned operations we obtain

$$q(r,\varphi) = \frac{\chi r\cos\varphi}{\lambda A a}\left\{\operatorname{erf}\left[\frac{D}{2\lambda}\left(1-\frac{r^2}{a^2}\right)\right]^{1/2} + \left[\frac{\pi}{2A\lambda}\left(1-\frac{r^2}{a^2}\right)\right]^{-1/2}\exp\left[-\frac{D}{2\lambda}\left(1-\frac{r^2}{a^2}\right)\right]\right\} \tag{6.8}$$

The expressions for the moments are evaluated by means of (4.14) for both versions of the solution and are, respectively

$$\begin{aligned} M_y = \frac{\pi\chi a^3}{A\lambda}\Bigg\{\Bigg[\frac{1}{4} + \left(\sqrt{AD}-\frac{1}{2}\right)p^{-2} + \left(\frac{5}{8}+AD-\frac{3}{2}\sqrt{AD}\right)p^{-4} + \Bigg(\frac{5}{4}\sqrt{AD} -\\ - AD - \frac{3}{8}\Bigg)p^{-6} + \left(\frac{3}{4}AD - \frac{3}{8}\sqrt{AD} - \frac{15}{64}\right)p^{-8}\Bigg]\operatorname{erf} p + \Bigg[\frac{1}{4} + \left(\sqrt{AD}-\frac{5}{8}\right)p^{-2} +\\ + \left(\frac{17}{16}+AD-\sqrt{AD}\right)p^{-4} + \left(\frac{15}{32}-\frac{3}{2}AD+\frac{3}{4}\sqrt{AD}\right)p^{-6}\Bigg]\frac{\exp(-p^2)}{p\sqrt{\pi}}\Bigg\} \end{aligned} \tag{6.9}$$

$$\begin{aligned} M_y = \frac{\pi\chi a^3}{4A\lambda}\Bigg\{[1 + (2\sqrt{AD}-1)2p^{-2} + (3-4\sqrt{AD})p^{-4}]\operatorname{erf}\frac{p}{\sqrt{2}} +\\ + 2[1-(3-4\sqrt{AD})p^{-2}]\frac{\exp(-{}^1/{}_2 p^2)}{p\sqrt{2\pi}}\Bigg\} \end{aligned} \tag{6.10}$$

Values of the quantity $M^* \approx M_y/\chi_a{}^3$, computed for comparison by (6.9) and (6.10) are shown on the right

$\lambda =$	1/8	1/4	1/2	
$M^* =$	13.94	7.72	4.75	(6.9)
$M^* =$	13.90	7.67	4.57	(6.10)

We see from the numerical values shown, that the formulas for M_y are asymptotically equivalent for small λ.

In conclusion, let us note that the results obtained here are completely applicable to the analogous contact problem for a layer when its lower boundary is connected rigidly to an nondeformable base. In the latter case only the values of the constants A and D in the approximation (2.17) change.

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TABLE

	λ	χ	$x/a=0$	0.4	0.6	0.8	0.95	ω/χ	$P/\chi a$
	1/2		3.95	3.97	4.02	4.27	5.89	1.59	8.91
aq_0/χ	1		2.05	2.09	2.18	2.48	3.93	1.13	5.03
	2		1.14	1.19	1.28	1.54	2.67	0.80	3.05
	1/2	1.00	3.95	3.97	4.02	4.27	5.89	1.59	8.91
aq^*/χ	1	0.93	1.90	1.94	2.02	2.30	3.65	1.05	4.67
	2	0.87	0.99	1.04	1.12	1.35	2.33	0.70	2.66
	1/2		3.97	3.94	3.90	4.00	5.74	1.58	8.71
aq/χ[4]	1		1.92	1.93	1.99	2.24	3.80	1.12	4.70
	2		0.97	1.01	1.10	1.38	2.56	0.79	2.75

BIBLIOGRAPHY

1. Aleksandrov, V.M. and Vorovich, I.I., Kontaktnye zadachi dlia uprugogo sloia maloi tolshchiny (Contact problems for an elastic layer of slight thickness) *PMM*, Vol. 28, No. 2 1964.

2. Aleksandrov, V.M., O priblizhennom reshenii odnogo tipa integral' nykh uravnenii (On the approximate solution of one type of integral equations). *PMM*, Vol. 26, No. 5 1962.

3. Aleksandrov, V.M., Nekotorye kontaktnye zadachi dlia uprugogo sloia (Some contact problems for an elastic layer). *PMM*, Vol. 27, No. 4 1963.

4. Aleksandrov, V.M. and Babeshko, V.A., Kontaktnye zadachi dlia uprugoi polosy maloi tolshchiny (Contact problems for an elastic strip of slight thickness). Izv. AN SSSR, Mekhanika, No. 2 1965.

5. Shtaerman, I. Ia., Kontaktnaia zadacha teorii uprugosti (Contact problem of elasticity theory). GOSTEKHIZDAT, 1949.

6. Krein, M.G., Ob odnom novom metode resheniia lineinykh integral'nykh uravnenii pervogo i vtorogo roda (On a new method of solving linear integral equations of the first and second kind). Doklady AN SSSR, Vol. 100, No. 3 1955.

7. Aleksandrov, V.M. and Aleksandrov, G.P., Kontaktnaia zadacha teorii uprugosti dlia uprugoi polosy (Contact problem of elasticity theory for an elastic strip). Materialy II Nauchn. konf. aspirantov. Izd. Rostovsk. Un-ta, 1960.

8. Aleksandrov, V.M. and Vorovich, I.I., O deistvii shtampa na uprugii sloi konechnoi tolshchiny (On the effect of a stamp on an elastic layer of finite thickness) *PMM* Vol. 24, No. 2 1960.

9. Vorovich, I.I. and Ustinov, IU. A., O davlenii shtampa na sloi konechnoi tolshchiny (On pressure of a stamp on a layer of finite thickness). *PMM*, Vol. 23, No. 3 1959.

10. Aleksandrov, V.M., K zadache o deistvii shtampa na uprugii sloi konechnoi tolshchiny (On the problem of the effect of a stamp on an elastic layer of finite thickness). Materialy III nauchn. konf. aspirantov. Izd. Rostovsk Un-ta, 1961.

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